SMOOTHING PARAMETER SELECTION AND ALPHA-STABLE P-ADIC TIME SIGNALS

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ABSTRACT

The estimation of the spectral density of stable p-adic signals is already done. Such estimation is based on smoothing the periodogram by using a spectral window. The convergence rate of this estimator depends on bandwidth of spectral window (called the smoothing parameter). The aim of this work is to give a technique for selecting the optimal parameter, i.e. the parameter that achieves the estimation with the best convergence rate.

For that purpose, we were inspired by the cross-validation method of finding the optimal parameter. This method minimizes the integrated square error estimate.

KEYWORDS

Spectral density, p-adic processes, stable random field.

1. INTRODUCTION

In this paper, we are interested in a family of random fields whose energy is infinite, namely the family of alpha stable random fields. These processes are considerably accurate models for many phenomena in several areas such as physics, biology, electronics and electricity, hydrology, economics, communications and radar applications... see [1]-[12]. The estimation of the spectral density of these processes is given in [13] when the time processes is continuous, and in [14]-[15] when the time processes is discrete. The paper [16] has extended this work to stable random fields with p-adic time. More specifically, considers a harmonized p-adic process:

$$X(t_1,t_2)=\int_{Q_p^2} e^{i<t_1\lambda_1+t_2\lambda_2>} dM(\lambda_1,\lambda_2);$$

$(t_1,t_2)\in Q_p^2$ whereQ_p^2 is the field of p-adic numbers and M is a alpha stable random measure with a control measure m. The paper [16] studied the case where the measure m is continuous with respect to the Haar measure:

$dm(x_1,x_2)=\Phi(x_1,x_2)d\mathcal{H}(x_1,x_2),\quad (x_1,x_2)\in Q_p^2$. The density function Φ is called the spectral density of the process X. The paper[16] gives a modified periodogram from the observation of the process on the ball U_n. Afterwards, this modified periodogramis smoothed by a spectral window to get an asymptotically unbiased consistent estimate of the spectral density. Thus, the

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rate convergence of this estimator depends on the bandwidth of spectral window called the smoothing parameter.

The aim of this work is to give a method of selection for obtaining the optimal smoothing parameter, which allows achieving the best convergence rate of estimator. This method is based on cross validation technique.

The choice of the p-adic numbers is motivated by widespread use in various areas.

In view of the fact that many questions in physics have been solved thanks to p-adic numbers. Moreover, the number of articles dealing with the p-adic numbers definitely proves its importance. There are other questions relatedto the string theory (related to the p-adic quantum domain) and other areas where hierarchically structured fractal behaviors exist (turbulence theory, dynamical systems, statistical physics, biology, see [17]-[21]. The 2-adic number represents a special case useful for computer design see [22]. The quantum mechanical state has been modeled by using p-adic statistics [23]-[25] focusing on the probabilities when the number of trials is infinitely large. A new asymptotic of Bernoulli's classical probabilities has been demonstrated by [26]. The work [27] developed p-adic probability theory for statistical information processes. The paper [28] is interested in symmetric stochastic integrals with respect to p-adic Brownian motion. The works [29]-[32] studied the properties of the trajectories of a p-adic Wiener process. They used the p-adic differentiation operator.

The spectral theory and Fourier transforms of p-adic time stationary processes were developed in [34] where the process $X(t) t \in Q_p$ where Q_p is in time in the field of p-adic numbers. It also gave an estimate of the spectral density via the construction of a periodogram inspired by real-time stationary processes. Another estimator was proposed in [35] from discrete-time observations $X(\tau_k)_{k\in Z}$ where $(\tau_k)_{k\in Z}$ where the instant observations are taken from Q_p, associated with a Poisson process.

The paper is organized as follows: Section 2 recalls the asymptotically unbiased and consistent estimate (smoothing the periodogram) and the results (propositions 2.1-2.5) shown in [34]. Section 3 shows that the estimator which tends to the spectral density in probability convergence mode (proposition 2.6), gives the cross validation criterion and shows that criterion gives the optimum smoothing parameter (theorems 3.1-3.3). Section 4 contains the concluding remarks, the potential applications and the open research problems.

2. PERIODOGRAM AND SPECTRAL DENSITY ESTIMATION

Consider a process $X = \{X_{t_1,t_2} / (t_1, t_2) \in Q_p^2\}$ where Q_p^2 is the field of p-adic numbers having the following integral representation

$X_{t_1,t_2} = \int_{Q_p^2} e^{i<t_1\lambda_1 + t_2\lambda_2>} dM(\lambda_1, \lambda_2) (1)$

$\forall (t_1, t_2) \in Q_p^2$ where M is a symmetric α stable $S\alpha S$ random measure with independent and isotropic increments. There exists a control measure m that is defined by:

$$m(A \times B) = [M(A \times B), M(A \times B)]_\alpha^{1/\alpha}.$$

Assume that the measure m is absolutely continuous with respect to Haar measure: $dm = \Phi(x_1, x_2) d\mathbb{H}(x_1, x_2)$ where $\mathbb{H}$ is Haar measure.

This density is estimated in [13] when the process has continuous real time, in [14]-[15] when the process and the random field have discrete time. The paper [16] studied the process when the time is p-adic. They give an asymptotically and consistent estimate of the density Φ called the spectral density of the process $X = \{X_{t_1,t_2}/(t_1,t_2) \in Q_p^2\}$. For that, they take the ball $U_n = \{(x,y) \in Q_{p;;}^2 |(x,y)|_p \leq p^n\}$ as the observation of the process. They consider the following periodogram: For $(\lambda_1,\lambda_2) \in Q_p^2$

$$d_n(\lambda_1,\lambda_2) = A_n Re \int_{U_n} e^{-i<t_1\lambda_1+t_2\lambda_2>} p^{-n} h(t_1 p^n, t_2 p^n) X(t_1,t_2) d\mathcal{H}((t_1,t_2)) \quad (2)$$

$$H(\lambda_1,\lambda_2) = \int_{Z_p^2} h(t_1,t_2) e^{-i<t_1\lambda_1+t_2\lambda_2>} d\mathcal{H}(t_1,t_2)$$

$$B_\alpha = \int_{Q_p^2} |\mathcal{H}(\lambda_1,\lambda_2)|^\alpha d\mathcal{H}(\lambda_1,\lambda_2) < +\infty$$

$$H_n(\lambda_1,\lambda_2) = \left(\frac{p^{2n}}{B_\alpha}\right)^{\frac{1}{\alpha}} H(p^{-n}\lambda_1, p^{-n}\lambda_2) \ = A_n H(p^{-n}\lambda_1, p^{-n}\lambda_2)$$

Therefore, $A_n = \left(\frac{p^{2n}}{B_\alpha}\right)^{\frac{1}{\alpha}}$.

$$\int_{Q_p^2} |H_n(\lambda_1,\lambda_2)|^\alpha d\mathcal{H}(\lambda_1,\lambda_2) =$$

$$\int_{Q_p^2} \frac{p^{2n}}{B_\alpha} |H(p^{-n}\lambda_1 p^{-n}\lambda_2)|^\alpha d\mathcal{H}(\lambda_1,\lambda_2) = \frac{p^{2n}}{B_\alpha} p^{-2n} \int_{Q_p^2} |H(v_1,v_2)|^\alpha d\mathcal{H}(v_1,v_2)$$

Since $|p^{-n}|_p = p^{+n}$, they obtain

$$\int_{Q_p^2} |H_n(\lambda_1,\lambda_2)|^\alpha d\mathcal{H}(\lambda_1,\lambda_2)=1$$

The following propositions 2.1-2.5 are proved in [16]

Proposition 2.1 Let

$\Psi_n(\lambda_1,\lambda_2) =^\Delta \int_{Q_p^2} |H_n(\lambda_1-u_1,\lambda_2-u_2)|^\alpha \Phi(u_1,u_2) d\mathcal{H}(u_1,u_2)$.If $\Phi$is a continuous and boundedfunction, then$B_\alpha\left(\Psi_n(\lambda_1,\lambda_2) - \Phi(\lambda_1,\lambda_2)\right)$ converges to zero as n tends to infinity.

Proposition 2.2 *Let $(\lambda_1,\lambda_2) \in Q_p^2$ the characteristic function of $d_n(\lambda_1,\lambda_2), E exp\{ir d_n(\lambda_1,\lambda_2)\}$, converges to $exp\{-C_\alpha |r|^\alpha \Phi(\lambda_1,\lambda_2)\}$.*

The periodogramis modified as follows:

$$I_n(\lambda_1,\lambda_2) = C_{q,\alpha} |d_n(\lambda_1,\lambda_2)|^q, \quad (3)$$

$where$0<q<2 and the normalization constant is given by $C_{(q,\alpha)} = \frac{D_q}{F_{q,\alpha} C_\alpha^{q/\alpha}}$ where $D_q = \int \frac{1-cos(u)}{|u|^{1+q}} du$ and $F_{q,\alpha} = \int \frac{1-\exp(-|u|^\alpha)}{|u|^{1+q}} du$ $C_\alpha = (\alpha\pi)^{-1} \int_0^\pi |\cos(\theta)|^\alpha d\theta$,

Propositions 2.3 *Let $(\lambda_1,\lambda_2) \in Q_p^2$, then $EI_n(\lambda_1,\lambda_2) = \left(\Psi_n(\lambda_1,\lambda_2)\right)^{\frac{q}{\alpha}}$ and $I_n(\lambda_1,\lambda_2)$ is an asymptotically unbiased estimator of the spectral density but not consistent*

$EI_n(\lambda_1,\lambda_2) - \left(\Phi(\lambda_1,\lambda_2)\right)^{q/\alpha}$=o(1) and $VarI_n(\lambda_1,\lambda_2) - V_{\alpha,q}\left(\Phi(\lambda_1,\lambda_2)\right)^2$, with $V_{\alpha,q} = \frac{C_{q,\alpha}^2}{C_{2q,\alpha}} - 1$.

In order to have an asymptotically and consistent estimate, we smooth the periodogram that was modified using a spectral window.

$$f_n(\lambda_1,\lambda_2) = \int_{Q_p^2} W_n(\lambda_1 - u_1, \lambda_2 - u_2) I_n(u_1,u_2) d\mathcal{H}(u_1,u_2)$$

Where $W_n(x_1,x_2) = |M_n|_p W(x_1 M_n, x_2 M_n)$ such that
$M_n \to \infty$; $\frac{M_n}{n} \to 0$; $|M_n|_p \to 0$ and $\frac{|M_n|_p}{|p^n|_p} \to \infty$. (4)

The function W is an even nonnegative function vanishing outside $[-1,1]^2$ and $\int_{Q_p^2} W(v_1,v_2) d\mathcal{H}(v_1;v_2) = 1$.

Proposition 2.4 *Let* $(\lambda_1,\lambda_2) \in Q_p^2$ *and* $Bias\left(f_n(\lambda_1,\lambda_2)\right) = E[f_n(\lambda_1,\lambda_2)] - \left(\Phi(\lambda_1,\lambda_2)\right)^{p/\alpha}$, *then* $Bias\left(f_n(\lambda_1,\lambda_2)\right) = o(1)$. *Moreover, if* Φ *verifies* $|\Phi(x_1,x_2) - \Phi(y_1,y_2)| \le cste|(x_1 - y_1, x_2 - y_2)|_p^{-k}$, *then,* $Bias\left(f_n(\lambda_1,\lambda_2)\right) = O\left(\frac{1}{|M_n|_p^{-\frac{kp}{\alpha}}}\right)$.

Proposition 2.5 *Let* (λ_1,λ_2) *be in* Q_p^2. *Assume that* $\Phi \in L^1_{Q_p^2}$. *Then* $Var\left(f_n(\lambda_1,\lambda_2)\right) = O(p^{-n} M_n^{-3n})$.

From propositions 2.4 and 2.5, we show in the following proposition that $(f_n(\lambda_1,\lambda_2))^{\frac{\alpha}{q}}$ converges to $\phi(\lambda_1,\lambda_2)$ in probability.

Proposition 2.6 Let λ_1,λ_2 *p-adic numbers such that* $\phi(\lambda_1,\lambda_2) > 0$. *Then,* $(f_n(\lambda_1,\lambda_2))^{\frac{\alpha}{q}}$ *converges in probability to* $\phi(\lambda_1,\lambda_2)$.

Proof We show that $f_n(\lambda_1,\lambda_2)$ converges in mean quadratic to $\phi(\lambda_1,\lambda_2)^{\frac{q}{\alpha}}$. $E\left|f_n(\lambda_1,\lambda_2) - \phi(\lambda_1,\lambda_2)^{\frac{q}{\alpha}}\right|^2 = Ef_n(\lambda_1,\lambda_2) - \phi(\lambda_1,\lambda_2)^{\frac{q}{\alpha}})^2 + Var(f_n(\lambda_1,\lambda_2)$. Then, from proposition 2.3, $E\left|f_n(\lambda_1,\lambda_2) - \phi(\lambda_1,\lambda_2)^{\frac{q}{\alpha}}\right|^2$ converges to zero. Thus, $(f_n(\lambda_1,\lambda_2))^{\frac{\alpha}{q}}$ converges to $\phi(\lambda_1,\lambda_2)$ in probability.

It is clear that the choice of M_n has an important role since the convergence speeds depend on this smoothing parameter. The papers [35]-[36] gave a criterion of choice of h in the one-dimensional case, they were restricted to the parametric case. The objective of this work is to give a criterion for the selection of these parameters by non-parametric methods. Let's note by $f(x_1,x_2) = (\Phi(x_1,x_2))^{\frac{q}{\alpha}}$ and $h = \frac{1}{M_n}$ the width of the two spectral windows. We are therefore looking for a criterion $CV(h)$ allowing us to select $h$to minimize the mean integrated square error (MISE), where

$$MISE(h) = \int \int E[f_n(x_1, x_2) - f(x_1, x_2)]^2 \rho(x_1, x_2) dx_1 dx_2, \quad (5)$$

ρ being a weight function that is assumed to be known and null outside of $[0,2\pi] \times [0,2\pi]$. Although $MISE(h)$ it is a good measure of the quality off_n, it can not help us to choose h, since it depends on the unknown function f. We will therefore try to estimate it. For this, we adopt the method of cross validation that has been proposed in [35]. Indeed, consider the integrated square error (ISE) defined by:

$$ISE(h) = \int \int [f_n(x_1, x_2) - f(x_1, x_2)]^2 \rho(x_1, x_2) dx_1 dx_2 = A - 2C + B$$

$$\text{where} A = \int_0^{2\pi} \int_0^{2\pi} f_n^2(x_1, x_2) \rho(x_1, x_2) dx_1 dx_2$$

$$C = \int_0^{2\pi} \int_0^{2\pi} f_n(x_1, x_2) f(x_1, x_2) \rho(x_1, x_2) dx_1 dx_2$$

$$B = \int_0^{2\pi} \int_0^{2\pi} f^2(x_1, x_2) \rho(x_1, x_2) dx_1 dx_2.$$

Since B is Independent of h, to choose$h$ minimizing $ISE(h)$ is to choose h minimiszing $A - 2C$. The term A is calculable since we know f_n, whereas, in the term C, we have$f$ unknown. Weproceed by the principle of “leave-out- I”.

3. Construction of the Cross-Validation Estimator

In this section, we will define the estimator and give some results in the form of a proposition or theorem, the proof of which is appended in the last section. Let $j, j' \in \{0,1, \dots, n-1\}$ such that $\frac{2\pi j}{n} \in U_n$and $\frac{2\pi j'}{n} \in U_n$. The construction of “leave-out- I" consists of finding an estimator $f_n^{j,j'}(\omega_j, \omega_{j'})$ that replace $f(w_j, w_{j'})$ in the expression of C and such that $I_n(\omega_j, \omega_{j'})$ and $f_n^{j,j'}(\omega_j, \omega_{j'})$ are asymptotically independent. Thus, we can estimate $C$by:$\frac{1}{\bar{n}^2} \sum_{j \in A_n} \sum_{j' \in A_n} f_n^{j,j'}(w_j, w_{j'}) I_n(w_j, w_{j'}) \rho(w_j, w_{j'})$ where $w_j = \frac{2\pi j}{n}, w_{j'} = \frac{2\pi j'}{n}, \bar{n} = \left[\frac{n-1}{2}\right]$ and $A_n = \left\{ j \in \{0,1, \dots, n-1\} such\ that \frac{2\pi j}{n} \in U_n \right\}$

$$f_n^{j,j'}(x_1, x_2) = \int_{U_n^2} I_n^{j,j'}(u_1, u_2) W_n(x_1 - u_1, x_2 - u_2) du_1 du_2, where$$

$$I_n^{j,j'}(u_1, u_2) = I_n(u_1, u_2) if \ (u_1, u_2) \notin B_{j,j'}$$

$$I_n^{j,j'}(u_1, u_2) = \theta_1(u_1, u_2) I_n(w_{j-1}, w_{j'-1}) + \theta_2(u_1, u_2) I_n(w_{j+1}, w_{j'-1}) + \theta_3(u_1, u_2) I_n(w_{j-1}, w_{j'+1}) + \theta_4(u_1, u_2) I_n(w_{j+1}, w_{j'+1}) \quad otherwise$$

$B_{j,j'} =]w_{j-1}, w_{j+1}[\times]w_{j'-1}, w_{j'+1}[$. The construction of $I_n^{j,j'}(u_1, u_2)$ where $(u_1, u_2) \in A_{j,j'}$ is done as if I_n was bi-linear. In this case,

$\theta_1(u_1, u_2) = \alpha\beta$; $\theta_2(u_1, u_2) = (1 - \alpha)\beta$; $\theta_3(u_1, u_2) = \alpha(1 - \beta)$ and
$\theta_4(u_1, u_2) = (1 - \alpha)(1 - \beta)$where$\alpha = \frac{u_1 - w_{j+1}}{w_{j-1} - w_{j+1}}$ and $\beta = \frac{u_2 - w_{j'+1}}{w_{j'-1} - w_{j'+1}}$.

The following proposition shows that $f_n^{j,j'}$ is an estimator asymptotically unbiased of the function f.

Theorem 3.1*For all*$(x_1, x_2) \in Q_p^2$

$$E\left[f_n^{j,j'}(x_1, x_2) - f_n(x_1, x_2)\right] = O\left(\frac{1}{n^2}\right).$$

From this result, we establish our criterion, noted CV "cross validation", defined by :

$$CV(h) = \quad CV_1(h) + \int_{U_n{}^2} f^2(u_1, u_2)\rho(u_1, u_2)du_1du_2$$

$$where\ \ CV_1(h) = \int_{U_n{}^2} f_n^2(u_1, u_2)\rho(u_1, u_2)du_1du_2 -$$

$$\frac{2}{n^2}\sum_{j\in A_n}\sum_{j'\in A_n} f_n^{j,j'}(w_j, w_{j'})I_n(w_j, w_{j'})\rho(w_j, w_{j'})$$

The widths of spectral windows will be chosen at the points $\hat{h}$ minimizing the criterion$CV(h)$:
$\hat{h}_1 = \operatorname{argmin}_h CV(h) = \operatorname{argmin}_h CV_1(h)$ (6)

Subsequently, to facilitate writing and to generalise without losing generality, we will take $\rho(u_1, u_2) = \frac{1}{2\pi}$ on U_n^2 and null outside.

Now, we establish results similar to those given in [37], concerning the estimation of the intensities of a punctual process. It is to show that, on average, when n is large enough, the criterion $CV(h)$ is approximately equal to the integrated quadratic error $ISE(h)$ and that the variance of $CV(h)$ is asymptotically zero. This allows us to confirm that the parameters $\hat{h}$ minimizing the criterion $CV(h)$are close to those that minimize the integral squared error (ISE) when n is large enough. These results are stated in the following theorem:

Theorem 3.2*We have*

$$|E\{CV(h) - ISE(h)\}| = \quad O\left(\frac{1}{n^2}\right).$$

$$var\{CV(h)\} = \quad O\left(\frac{1}{n^2h^2}\right)$$

Thus, since

$$E\{[CV(h) - MISE(h)]^2\} =$$

$$var\{CV(h)\} + [E\{CV(h) - MISE(h)\}]^2 = \quad O\left(\frac{1}{n^2h^2}\right).$$

The widths of the spectral windows $\hat{h}_1$ and $\hat{h}_2$ obtained by cross validation, defined in (6), are asymptotically optimal, i.e. the integrated square error at $\hat{h}$ converges in probability to the small integrated square error.

Theorem 3.3*The width of the spectral windows $\hat{h}$ obtained by cross validation are asymptotically optimal:*

$$\frac{ISE(\hat{h})}{ISE(\hat{\hat{h}})} \longrightarrow 1 \qquad in\ probability, \qquad where$$

$$\hat{h} = \underset{h}{\operatorname{argmin}} CV(h) \quad and \quad (\hat{\hat{h}}) = \underset{h}{\operatorname{argmin}} ISE(h).$$

To show this result, we use the similar technique used in [38].

4. CONCLUSION

In this work, we propose a method to find the smoothing parameter that optimizes the estimation of the spectral density for a p-adic time alpha-sand process. The approach is based on the cross-validation technique, which has already given very good results in other fields. This kind of process finds its applications to model phenomena, which have an infinite energy such as:

Dynamic images of an agricultural field taken by drones, to identify weeds. These images can have a great variance, in particular when the images are disturbed by climatic conditions. - The amount of certain microorganisms in the soil varies in a significant way that can be modelled by alpha stable random field. The measurement of microorganisms can be taken with a constant error and with some jumps when encountering rocks in the ground. This work could be completed by the study of a more general case when the measurement is mixed and when the observations are taken with a constant error.

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